# 2015 John O'Bryan Mathematical Competition
# 5-person Team Test

Abbreviated Instructions: Answer each of the following questions **using separate sheet(s) of paper for each numbered problem**. Place your team code in the upper right corner of each page that will be turned in (failure to do this will result in no score). Place problem numbers in the upper left corner. Problems are equally weighted; **teams must show complete solutions (not just answers) to receive credit**. More specific instructions are read verbally at the beginning of the test.

1. In the game of tic-tac-toe, X's and O's are placed into a three-by-three grid. A **winning line** of X's (or O's) is formed by three consecutive X's (or O's) found horizontally, vertically, or diagonally in the grid. Suppose that six X's and three O's are randomly placed on the tic-tac-toe grid.

    a. Show that there will always be a **winning line**.

    b. Show that either the O's win by having the only **winning line**, or that the X's win by having more **winning lines** than the O's.

2. Solve each of the following:

    a. If $26i-(a+bi)^2=c$ where $a$, $b$, and $c$ are integers and $i^2=-1$, find all possible ordered triples ($a$, $b$, $c$).

    b. For what values of $x$ is $\sqrt{\left(x+\sqrt{2x-1}\right)}+\sqrt{\left(x-\sqrt{2x-1}\right)}=A$. Consider the possibilities $A=\sqrt{2}$ and $A=1$. Only real-valued arguments are permitted.

3. Consider a standard 8 by 8 checkerboard. There are a total of 64 one-inch squares on the board. Using only these squares as they are arranged in eight rows and eight columns:

    a. How many 2 x 2 squares can be formed?

    b. How many total squares can be formed?

    c. How many total rectangles can be formed?

4. An urn contains 4 green balls and 6 blue balls. A second urn contains 16 green balls and $N$ blue balls.

    a. If two balls are chosen at random from the first urn, what is the probability that there is one of each color?

    b. If two balls are chosen at random from the second urn, then the probability that they are different colors is $\frac{32}{63}$. Find all possible $N$.

5. Square ABCD has side length 13. Points E and F are exterior to the square such that BE = DF = 5 and AE = CF = 12. Find $EF^2$.

6. Suppose that $a$ and $b$ must be integers with $2 \le a \le 100$ and $2 \le b \le 100$. Determine the **number** of ordered pairs $(a,b)$ such that $\log_a b + 6\log_b a = 5$

**Solutions to these problems will be posted after the contest to: http://math.nku.edu/job**

**1.** Six X' s and 3 O's are randomly placed on a tic-tac-toe board.

a. Show that there will always be a three in a row.

b. Show that either O wins by having the only three-in-a-row, or that X wins by having two three-in-a-rows to one for O, or wins 1or 2 to 0.. (In particular, there can never be a tie.)

a. *If neither O nor X has three-in-a-row, then O must have a marker in every row and column. After symmetries there are only two possibilities for where the O is in the first column:*

*In the top, left corner. Then there is only one possible way to place the other two O's and not have three-in-a-row and in that case the X's have three-in-a-row.*

*In the middle row. Then there is only one way (up to symmetries to place the other two O's and in that case X has three-in-a-row.*

*So if the O's do not have three-in-a-row, then the X's must.*

b. *If the three O's are on a diagonal, then X has no three-in-a-rows. O-1, X-0*
*If the three O's form a row (or a column) then the X's fill the other two rows (cols). O-1, X-2*
*If the O's do not have three-in-a-row, then the xs's always do. O-0, X- 1 (or more?)*

**2.** Each of the followng is worth 5 points:

**2.1.** If 26 i - (a+i $b)^2$= c, where a,b,and c, are all integers, and $i^2 = -1$, find a,b, and c.

*Simplify to 2 i (13-a b) $+(b^2 - a^2) = c$. So a b =13, with a and b integers. So a=13, b=1, or a=-13, b=-1 giving c to be -168; or a=1, b=13, or a=-1 b=-13, giving c to be +168.*

**2.2.** For what values of x is $\sqrt{\left(x+\sqrt{2x-1}\right)} + \sqrt{\left(x-\sqrt{2x-1}\right)}$ =A, given that

a) $A = \sqrt{2}$ ?

$b) A = 1$ ?

(Only non – negative real numbers are admitted for square roots.)

*$A^2 = 4x - 2$, just working through the expansion algebraically. So for $A = \sqrt{2}$, you get $x = 1$, which can be checked. For A=1 you get x=3/4, and this does not check - the first term, $\sqrt{\left(x+\sqrt{2x-1}\right)}$, is already larger than 1.*

**3.** Consider a standard 8 by 8 checkerboard. There are a total of 64 1 by 1 squares on the board.

a. How many 2 x 2 squares?

b. How many total squares?

c. How many total rectangles?

a. *There are 7 places left-right or up-down for one corner of the square, so $7^2$=49 2x2 squares.*

b. *similar to a: $8^2 + 7^2 + \ldots + 1^2$= 204 total squares*

c. *For the horizontal sides of the square we choose 2 of the 9 possible sides. Similarly vertically: ( 9 choose 2)$^2$= $((9 \times 8)/2)^\wedge 2 = 36^2 = 1296$ rectangles.*

**4.** An urn contains 4 green balls and 6 blue balls. A second urn contains 16 green balls and N blue balls.
a. If two balls are chosen from the first urn, what is the probability there is one of each color?
b. If two balls are chosen at random from the second urn, then the probabilty that they are different colors is $\frac{32}{63}$. Find all possibilities for N.

a. $\frac{4 \times 6}{10\ \text{Choose}\ 2}$, *where 10Choose2 = $(10 \times 9)/2 = 45$, so the probabilty is $\frac{24}{45}$.*

b. $\frac{16 \times N}{(16+N)\ \text{Choose}\ 2}$,. *This time we have (16+N) Choose 2 = $\frac{(16+N)(15+N)}{2}$. So we have*
$(16 N)/(((16 + N)(15 + N))/2) = \frac{32}{63}$, *or $N^2 + 31 N + 15 \times 16 = 63 N$ or $N = 12, 20$*

**5.** Square ABCD has side length 13, and points E and F are exterior to the square such that BE = DF = 5 and AE = CF = 12..Find $EF^2$.

*Extend to the obvious circumscribing square. This is a square with side length 17 and so diagonal length 17 sqrt(2).=578*

**6.** Determine the number of ordered pairs (a, b) of integers such that $\log_a b + 6 \log_b a = 5$, where $2 \leq a \leq 100$ and $2 \leq b \leq 100$.

*$\log_a b = 1/\log_b a$, so we have $\log_a b + 6 / \log_a b = 5$, multiply thru by $\log_a b$ to get $X^\wedge 2 - 5 + 6 = 0$, where $X = \log_a b$. This quickly reduces to $\log_a b = 2, 3$. then $b = a^3$ or $b = a^2$. So points (2,4),(3,9),(4,16),..(10,100) and (2,8),(3,27), (4, 64).*